First Passage Time Distribution and Number of Returns for Ultrametric Random Walk

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Abstract

In this paper, we consider a homogeneous Markov process $\xi(t;\omega)$ on an ultrametric space Q_p , with distribution density $f(x,t)$, $x \in Q_p$, $t \in R_+$, satisfying the ultrametric diffusion equation $\frac{\partial}{\partial t} f(x,t) = -D_x^{\alpha} f(x,t)$. We construct and examine a random variable $\tau_{Z_p}(\omega)$ that has the meaning the first passage times. Also, we obtain a formula for the mean number of returns on the interval $(0,t]$ and give its asymptotic estimates for large t.

Introduction

Ultrametric random processes and their physical and biological applications have recently been attracting much attention, especially in connection with modelling the dynamics and evolution of complex systems characterized by multidimensional rugged energy landscapes (fitness landscapes) with a huge number of local minima (see, for instance, [1]–[7]). It is clear that a description of the dynamics on such landscapes requires adequate approximations. As shown recently, a reasonable approximation for the dynamics of some biological systems (in particular, proteins) can be chosen in the form of random "jumps" between local minima of a landscape, under the assumption that the only key factor is the maximal activation barrier on the landscape that separates these local minima [8]. In this case, the local minima are clustered in "basins" of minima hierarchically embedded in one another. Accordingly, the dynamics of such a system is described in terms of random "jumps" between the basins. As shown in recent publications [4]–[6] such approximations can be naturally described in terms of ultrametric random processes, and it turns out that the p-adic pseudodifferential equation of ultrametric diffusion (introduced in [9] and called there *the equation of Brownian motion on the p-adic line*) gives an adequate description of protein dynamics [5],[10].

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Clearly, a physically meaningful application of the ultrametric diffusion equation requires an answer to some questions pertaining to the description of experimentally observable quantities. In this connection, it should be mentioned that the observable quantity in a real experiment corresponds to some specific degrees of freedom (determined by the adopted method of measurement) of a complex system with many degrees of freedom. In some cases, the observable quantity corresponds to a characteristic of the system averaged over all its states [5], in other situations, it is determined by some specific states [10]. In a situation of this kind, there arise classical probability problems for ultrametric diffusion such as the problem of the first passage time distribution and the problem of the number of returns.

In this paper, we consider a homogeneous Markov process $\xi(t;\omega)$ on an ultrametric space Q_p (ultrametric random walk), with distribution density $f(x,t)$, $x \in Q_p$, $t \in R_+$, satisfying the equation

$$\frac{\partial}{\partial t} f(x,t) = -D_x^\alpha f\,(x,t)\,,$$

usually called *the ultrametric diffusion equation* (for definitions and notation see below). We consider a specific random process $\xi(t;\omega)$, namely, that whose distribution density satisfies the Cauchy problem for the ultrametric diffusion equation with the initial density in a domain $Z_p \subset Q_p$.

Our aim is to construct and examine a random variable $\tau_{Z_p}(\omega)$ that has the meaning the first time instant when the trajectories of the random process $\xi(t;\omega)$ return to the domain Z_p. To study this problem, we first prove that the distribution density of $\tau_{Z_p}(\omega)$, denoted by $f(t)$, satisfies a nonhomogeneous Volterra equation, then we construct a solution of that equation and examine its properties. On the other hand, we show that the first passage time distribution density can be represented as a functional of a density function which is a solution of the ultrametric diffusion equation with the absorbing region Z_p. It is shown that these two approaches are equivalent. In the last part of the paper, we consider the problem of the number of returns to the domain Z_p on the time interval $(0,t]$. We obtain a recurrent equation for the probability $q^{(m)}(t)$ of the m-th return on the time interval $(0,t]$, as well as a recurrent equation for the probability $h^{(m)}(t)$ of precisely m returns on the interval $(0,t]$. We study the properties of the functions $q^{(m)}(t)$, $h^{(m)}(t)$ and obtain a formula for the mean number of returns on the interval $(0,t]$ and give its asymptotic estimates for large t.

Section 1 contains some basic facts from p-adic analysis and the theory of random processes. These facts are used for the introduction of the necessary notation and definitions. In Section 2, we consider the first passage problem for ultrametric random walk. In Section 3, we introduce and examine a p-adic analogue of the diffusion equation with an absorbing region for the first passage problem. Section 4 is dedicated to the problem of the number of returns for ultrametric diffusion.

1 Elements of p-Adic Analysis and the Theory of Random Processes

Let Q be the field of rational numbers and $p \in Q$ a fixed prime. Any rational number $x \neq 0$ can be uniquely represented in the form

$$x = p^{\gamma}\frac{a}{b},$$

where $a,\ b,\ \gamma \in Z$ are integers; a and b are coprime positive integers indivisible by p. The *p-adic norm* $|x|_p$ of $x \in Q$ is defined by the relations $|x|_p = p^{-\gamma}$, $|0|_p = 0$. The completion of the field of rational numbers Q with respect to the p-adic norm is denoted by Q_p and is called *the field of p-adic numbers.* The set Q_p endowed with the metric $\rho(x,y) = |x-y|_p$ is an ultrametric space which is complete, separable, totally disconnected, and locally compact. There is a unique (to within a coefficient) Haar measure $d_p x$ on Q_p which is translation-invariant: $d_p\,(x+a) = d_p x$. We normalize this measure by the condition

$$\int_{Z_p} d_p x = 1,$$

where $Z_p = \left\{x \in Q_p : \quad |x|_p \leq 1\right\}$ is the ring p-adic integers. There is only one measure $d_p x$ satisfying the above condition.

We introduce a class W^{α} $(\alpha \geq 0)$ of complex-valued functions $\varphi(x)$ on Q_p satisfying the following conditions:

(i) $|\varphi(x)| \leq C\left(1 + |x|_p^{\alpha}\right)$, where C is a constant;

(ii) there is an integer $N = N(\varphi) > 0$ such that $\varphi\,(x + x') = \varphi\,(x)$ for any $x \in Q_p$ and any $x' \in Q_p$ such that $|x'|_p \leq p^{-N}$.

A function $\varphi(x)$ satisfying these two conditions is called *locally constant*, and $N(\varphi)$ is called its *index of locally constancy.* For a function φ that additionally depends on a real parameter t, we say that $\varphi \in W^{\alpha}$ *uniformly in* t, if the constant C and the index N do not depend on t .

Functions in W^0 with compact support are called *test functions* (or *Bruhat–Schwartz functions*). The set of all test functions is denoted by D, and the set of distributions on D is denoted by D'.

Let χ be a normalized additive character of the field Q_p. Then $\chi \in W^0$. The *Fourier transform* of a function $\varphi\,(x) \in L^1\,(Q_p, d_p x)$ is defined by

$$\tilde{\varphi}\,(k) = \int_{Q_p} \chi\,(kx)\,\varphi\,(x)\,d_p x, \quad k \in Q_p. \tag{1.1}$$

For $\tilde{\varphi}\,(k) \in L^1\,(Q_p, d_p k)$, the *inverse Fourier transform* is defined by

$$\varphi\,(x) = \int_{Q_p} \chi\,(-kx)\,\tilde{\varphi}\,(k)\,d_p k, \quad x \in Q_p. \tag{1.2}$$

The operator D_x^{α} (*the Vladimirov pseudodifferential operator* [9]), $\alpha > 0$, is defined on functions $\varphi \in W^{\beta}$, $0 \leq \beta < \alpha$, by the formula

$$D_x^{\alpha}\varphi\,(x) = -\frac{1}{\Gamma\,(-\alpha)}\int_{Q_p} d_p y\frac{\varphi\,(y) - \varphi\,(x)}{|x-y|_p^{\alpha+1}}\,, \tag{1.3}$$

where $\Gamma_p(-\alpha) = \frac{1-p^{-\alpha-1}}{1-p^{\alpha}}$ is the p-adic analogue of the gamma-function.

Below, we consider random processes over the field Q_p. According to the Kolmogorov axioms , a *measurable space* is a pair $\{\Omega, \Sigma\}$, where Ω is a set and Σ is a σ-algebra of subsets of Ω. A *probability space* is a triplet $\{\Omega, \Sigma, \mathrm{P}\}$, where $\{\Omega, \Sigma\}$ is a measurable space and P is a countably additive nonnegative measure on Σ such that $\mathrm{P}(\Omega) = 1$. An element $A \in \Sigma$ is called an *event*, and the measure $\mathrm{P}(A)$ is called the *probability of the event* A. Let $\{Y, \mathrm{B}\ \}$ be a measurable space. A mapping $\xi : \Omega \to Y$ is called $\Sigma|\mathrm{B}$-*measurable*, if $\xi^{-1}(\mathrm{B}) \subset \Sigma$. A $\Sigma|\mathrm{B}$-measurable mapping ξ is called *a random variable with values in* Y and is denoted by $\xi = \xi(\omega)$. Such a function $\xi(\omega)$ induces a probability measure $P_\xi(B) = \mathrm{P}\{\xi^{-1}(B)\}$ on sets $B \in \mathrm{B}$. The function $P_\xi(B)$ is called the *distribution function* of the random variable ξ.

A *random mapping* of a set T into a measurable space $\{Y, \mathrm{B}\}$ is defined as a mapping $\xi(t, \omega) : T \times \Omega \to Y$ such that for any fixed $t \in T$ it is a measurable mapping from (Ω, Σ) to $\{Y, \mathrm{B}\}$, i.e., for any $B \in \mathrm{B}$, we have

$$\{\omega \in \Omega : \xi(t, \omega) \in B\} \in \Sigma.$$

If the parameter t is interpreted as time, a random mapping is called a *random process.*

Let $Y \equiv Q_p$, $T \equiv R^1_+$. As a probability space one can take $\Omega \equiv Q_p$ and $\Sigma \equiv \mathrm{B}$, where B is the σ-algebra of all measurable subsets of Q_p. To define a Markov process on Q_p homogeneous with respect to time, it suffices to define its distribution density function $f(x, t)$ and the transition density $f(y, t|x)$ satisfying the following conditions:

1. $f(x, t)$ is B-measurable in $x \in Q_p$ for any t;
2. $\int_{Q_p} f(x, t)\, d_p x = 1$;
3. $f(y, t|x) \geq 0$ for any $x \in Q_p$, $y \in Q_p$, and $t > 0$;
4. $f(y, t|x)$ is $\mathrm{B} \times \mathrm{B}$-measurable in x, y for any $t > 0$;
5. $\int_{Q_p} f(y, t|x) d_p y \leq 1$ for any $x \in Q_p$ and $t \geq 0$;
6. for any $x \in Q_p$, $y \in Q_p$, $s \geq 0$, and $t \geq 0$, the Chapman–Kolmogorov condition holds:
$$f(z, t + s|x) = \int_{Q_p} f(z, t|y) f(y, s|x) d_p y; \tag{1.4}$$
7. for any $x \in Q_p$, $s \geq 0$, and $t \geq 0$, the compatibility condition holds:
$$f(z, t + s|x) = \int_{Q_p} f(z, t|y) f(y, s) d_p y. \tag{1.5}$$

In this case, $f(x, t)$ defines a one-point distribution function for the random process:

$$P(B, t) = \int_B f(x, t)\, d_p x,$$

and $f(y, t|x)$ defines the transition function for the homogeneous Markov process:

$$P(B, t|x) = \begin{cases} \displaystyle\int_B f(y, t|x) d_p x, & t > 0,\ x \in Q_p,\ B \in \mathrm{B}, \\ I_B(x), & t = 0, \end{cases}$$

where $I_B(x)$ is the characteristic function of the set B.

2 The First Passage Problem

Consider a homogeneous Markov process $\xi(t,\omega) : R^1_+ \times \Omega \to Q_p$ with the transition density

$$f(y,t|x) \equiv f(y-x,t) = \int_{Q_p} \exp(-|k|_p^\alpha t)\chi(k(y-x))\, d_pk. \tag{2.1}$$

The function $f(y-x,t)$ satisfies the Markovian conditions

$$\begin{aligned} &f(x,t) > 0, \quad \int_{Q_p} f(x,t)d_px = 1, \\ &f(x,t) \to \delta(x) \quad \text{in} \quad D' \quad \text{as} \quad t \to 0+\,, \\ &\int_{Q_p} f(x-y,t)f(y,t')d_py = f(x,t+t'). \end{aligned}$$

The function $f(y-x,t)$ of the form (2.1) is a fundamental solution of the ultrametric diffusion equation

$$\frac{\partial}{\partial t} f(x,t) = -D_x^\alpha f(x,t)\,. \tag{2.2}$$

This random process was introduced in [9] as a p-adic analogue of random walk (on the p-adic line), and equation (2.2) was interpreted as a p-adic analogue of the diffusion equation, although the operator D_x^α is nonlocal and its correspondence to the Laplace operator is problematic. Note that in contrast to Wiener processes, the p-adic random walk $\xi(t,\omega)$ admits no continuous trajectories, since Q_p is a totally disconnected topological space. The support of $\xi(t,\omega)$ belongs to the class of functions without discontinuities of the second kind (see, for instance, [9]). The operator D_x^α can be interpreted in more clear physical terms, if (2.2) is regarded as a kinetic equation [4]–[6], which is justified in view of the integral representation (1.3) of the pseudodifferential operator D_x^α.

One of the classical problems of random walk on the real line is that of finding the distribution function of the random variable describing the first time instant when the wandering particle returns to the origin. Consider a similar problem for the p-adic random walk $\xi(t,\omega)$ defined above.

Let the evolution of the distribution density function $\varphi(x,t)$ of the random process $\xi(t,\omega)$ be described by the Cauchy problem for the ultrametric diffusion equation

$$\frac{\partial}{\partial t}\varphi(x,t) = -\frac{1}{\Gamma_p(-\alpha)} \int_{Q_p} d_py \frac{\varphi(y,t)-\varphi(x,t)}{|y-x|_p^{\alpha+1}}, \tag{2.3}$$

with the initial condition

$$\varphi(x,0) = \Omega(|x|_p), \tag{2.4}$$

where $\Omega(|x|_p) = \begin{cases} 1, & |x|_p \le 1, \\ 0, & |x|_p > 1 \end{cases}$ is the characteristic function of the domain Z_p.

Definition. The random variable $\tau_{Z_p}(\omega) : \Omega \to R^1_+$ defined by the relation

$$\tau_{Z_p}(\omega) = \inf\left\{ t > 0 : |\xi(t,\omega)|_p \le 1,\ if\ \exists\, t' :\ |\xi(t',\omega)| > 1\,,\ 0 < t' < t \right\}$$

is called the *first passage time* of a trajectory of the random process $\xi(t,\omega)$ entering the domain Z_p (i.e., the first instant when it returns to Z_p).

The initial condition (2.4) obviously implies that

$$\mathrm{P}\{\omega \in \Omega : |\xi\,(0,\omega)| \leq 1\} = 1.$$

Theorem 1. *The distribution density function $f\,(t)$ of the random variable $\tau_{Z_p}\,(\omega)$ satisfies the nonhomogeneous Volterra equation*

$$g(t) = \int_0^t g(t-\tau)f(\tau)d\tau + f(t)$$

with

$$g(t) = -\frac{1}{\Gamma_p(-\alpha)}\int_{Q_p\setminus Z_p}\frac{\varphi(x,t)}{|x|_p^{\alpha+1}}dx.$$

Proof. Consider the event $A\,(t_i,t_j)$ that consists in that a particle staying in the domain $Q_p \backslash Z_p$ goes back to the domain Z_p at a time belonging to the interval $(t_i,t_j]$ (under the condition that at $t=0$ the particle stays in Z_p):

$$\begin{aligned} &A\,(t_i,t_j) = \\ &\quad = \{\omega \in \Omega : \quad \exists t \in (t_i,t_j],\ \lim_{t'\to t-0}\xi\,(t',\omega) \notin Z_p,\ \lim_{t'\to t+0}\xi\,(t',\omega) \in Z_p |\ \xi\,(0,\omega) \in Z_p\}. \end{aligned}$$

Consider also the event $B\,(t_i,t_j)$ that consists in that a particle staying in the domain $Q_p \backslash Z_p$ goes back to the domain Z_p for the first time at an instant belonging to the interval $(t_i,t_j]$:

$$B\,(t_i,t_j) = \{\omega \in \Omega : t_i < \tau_{Z_p}\,(\omega) \leq t_j\}.$$

Let us divide the interval $(0,t\,]$ into n parts:

$$0 \equiv t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n \equiv t.$$

We obviously have $A\,(t_{n-1},t_n) \subset \bigcup_{i=1}^n B\,(t_{i-1},t_i)$. Since $A\,(t_{n-1},t_n)\bigcap B\,(t_{n-1},t_n) = B\,(t_{n-1},t_n)$, it follows that

$$\begin{aligned} A\,(t_{n-1},t_n) &= A\,(t_{n-1},t_n)\bigcap\left(\bigcup_{i=1}^n B\,(t_{i-1},t_i)\right) = \\ &= \bigcup_{i=1}^n\Big(A\,(t_{n-1},t_n)\bigcap B\,(t_{i-1},t_i)\Big) = \\ &= \left\{\bigcup_{i=1}^{n-1}\Big(A\,(t_{n-1},t_n)\bigcap B\,(t_{i-1},t_i)\Big)\right\}\bigcup B\,(t_{n-1},t_n) \qquad (2.5) \end{aligned}$$

Let $\mathrm{P}\,\{A\,(t_{i-1},t_i)\}$ and $\mathrm{P}\,\{B\,(t_{i-1},t_i)\}$ be the probabilities of the events $A\,(t_{i-1},t_i)$ and $B\,(t_{i-1},t_i)$, respectively. Taking into account (2.5) and the incompatibility of the events

$B\left(t_{i-1},t_i\right)$, we can write

$$
\begin{aligned}
&\mathrm{P}\left\{A\left(t_{n-1},t_n\right)\right\}= && (2.6)\\
&=\sum_{i=1}^{n-1}\mathrm{P}\left\{A\left(t_{n-1},t_n\right)\bigcap B\left(t_{i-1},t_i\right)\right\}+\mathrm{P}\left\{B\left(t_{n-1},t_n\right)\right\}=\\
&=\sum_{i=1}^{n-1}\mathrm{P}\left\{A\left(t_{n-1},t_n\right)|B\left(t_{i-1},t_i\right)\right\}\mathrm{P}\left\{B\left(t_{i-1},t_i\right)\right\}+\mathrm{P}\left\{B\left(t_{n-1},t_n\right)\right\}=\\
&=\sum_{i=1}^{n-1}\left\{\mathrm{P}\left\{A\left(t_{n-1}-t_i,t_n-t_i\right)\right\}+\varepsilon(t_i-t_{i-1})\right\}\mathrm{P}\left\{B\left(t_{i-1},t_i\right)\right\}+\mathrm{P}\left\{B\left(t_{n-1},t_n\right)\right\}.
\end{aligned}
$$

Here, we have used the relation

$$\mathrm{P}\left\{A\left(t_{n-1},t_n\right)|B\left(t_{i-1},t_i\right)\right\}=\mathrm{P}\left\{A\left(t_{n-1}-t_i,t_n-t_i\right)\right\}+\varepsilon(t_i-t_{i-1}),$$

where $\varepsilon(t_i-t_{i-1})\to 0$ as $t_i-t_{i-1}\to 0$. On the other hand, the probability $\mathrm{P}\left\{A\left(t_{n-1},t_n\right)\right\}$ is determined by the solution $\varphi(x,t)$ of the Cauchy problem for the ultrametric diffusion equation (2.3) with the initial condition (2.4) and has the form

$$\mathrm{P}\left\{A\left(t_{n-1},t_n\right)\right\}=g(t_n)\left(t_n-t_{n-1}\right)+o\left(t_n-t_{n-1}\right), \tag{2.7}$$

where $g(t)$ is defined by

$$g(t)=-\frac{1}{\Gamma_p(-\alpha)}\int_{Q_p\setminus Z_p}\frac{\varphi(x,t)}{|x|_p^{\alpha+1}}dx. \tag{2.8}$$

The function $g(t)$ is interpreted as the density of the probability to go back to the domain Z_p at time t. Similarly, the probability $P\left\{B\left(t_{i-1},t_i\right)\right\}$ of the first passage to the domain Z_p on the time interval $(t_{i-1},t_i]$ for the same random process can be represented in the form

$$\mathrm{P}\left\{B\left(t_{i-1},t_i\right)\right\}=f(t_i)\left(t_i-t_{i-1}\right)+o\left(t_i-t_{i-1}\right), \tag{2.9}$$

where $f(t_i)$ is the sought density of the probability of the first passage to the domain Z_p at time t_i. Now, substituting (2.7) and (2.9) into (2.6) and passing to the limit as $\max_{i=1,\ldots,n}\left\{(t_i-t_{i-1})\right\}\to 0$, we obtain a nonhomogeneous Volterra equation of convolution type,

$$g(t)=\int_0^t g(t-\tau)f(\tau)d\tau+f(t). \tag{2.10}$$

Note that $g(t)$ is a continuous function, and therefore, equation (2.10) has a unique solution in the class of continuous functions (see, for instance, [11]). It is easy to check that $g(t)$ is a function with a finite growth exponent for $t\geq 0$, and therefore, $f(t)$ has a finite growth exponent for $t\geq 0$ and there exist Laplace transforms of the functions $g(t)$, $f(t)$ denoted by $G(s)$, $F(s)$, respectively. Passing to the Laplace transforms in (2.10), it is easy to find that

$$F(s)=\frac{G(s)}{1+G(s)}. \tag{2.11}$$

Let us calculate $G(s)$. Substituting the solution of the Cauchy problem (2.3)–(2.4), which has the form (see (2.1), (2.2))

$$\varphi(x,t) = \int_{Q_p} \Omega\left(|k|\right) \exp\left[- |k|_p^{\alpha} t\right] \chi\left(-kx\right) d_p k,$$

into (2.8), integrating the result in x, and then passing to the Laplace transforms in t, we get

$$G(s) = \int_{Q_p} \Omega(|k|_p) \frac{B_\alpha - |k|_p^\alpha}{s + |k|_p^\alpha} d_p k = (B_\alpha + s) J(s) - 1, \tag{2.12}$$

where

$$B_\alpha = \frac{(1-p^{-1})}{1-p^{-\alpha-1}}, \quad p^{-\alpha} < B_\alpha < 1,$$

$$J(s) = \left(1 - p^{-1}\right) \sum_{n=0}^{\infty} p^{-n} \frac{1}{s + p^{-\alpha n}} \, . \tag{2.13}$$

Substituting (2.12) into (2.11), we obtain the Laplace transform of the desired function:

$$F(s) = 1 - \frac{1}{\left(B_\alpha + s\right) J(s)} \, . \tag{2.14}$$

The function $F(s)$ is analytic in the domain $\operatorname{Re} s > 0$ and tends to zero as $|s| \to \infty$, uniformly with respect to $\arg s$. The function $F(s)$ is the Laplace transform of the function $f(t)$ with zero growth exponent: $|f(t)| < M$. Now, it is not difficult to show that $f(t)$ has the following properties:

1. For $\alpha \geq 1$, we have $\int_0^\infty f(t)dt = F(0) = 1$, which means that for $\alpha \geq 1$ the particle is sure to return to the initial region, and therefore, on an infinite time interval will go back to that region infinitely many times. In this case, however, there is no finite mean waiting time for the first passage:

$$\left\langle \tau_{Z_p} \right\rangle = \int_0^\infty t f\left(t\right) dt = - \lim_{\substack{s \to 0 \\ \operatorname{Re} s > 0}} \frac{d}{ds} F(s) \to +\infty \, .$$

2. For $0 < \alpha < 1$, we have $\int_0^\infty f(t)dt = F(0) = \frac{p}{p^\alpha} \left(\frac{p^\alpha - 1}{p-1}\right)^2 \equiv \; C_\alpha < 1$. This means that for small α, there exist trajectories of the unltrametric random walk that abandon the initial region never to go back. Note that for the real-valued Brownian motion the return property of its trajectories is missing only if the dimension of the space is greater than two.

Consider more closely the function $F(s)$. Clearly, it has simple poles at $s = -\lambda_k$, $k = 0, 1, 2, \ldots$, which are simple roots of the equation $J(s) = 0$, and $s = -B_\alpha \equiv -\lambda_{-1}$. From (2.13), it is easy to see that the values λ_k belong to the interval $p^{-\alpha(k+1)} < \lambda_k < p^{-\alpha k}$. The point $s = 0$ is essentially singular and is a limit point of the poles. The function $F(s)$ is non-meromorphic on the complex plane, and this is an obstacle to the application of the residue theory for the calculation of the inverse Laplace transforms. To overcome this obstacle, we first prove the following result.

Lemma 1. *The function $F(s)$ can be represented as an infinite sum of terms with simple poles at the points $s = -\lambda_k$, $k = -1, 0, 1, 2, \ldots$, namely,*

$$F(s) = \sum_{k=-1}^{\infty} \frac{b_k}{s + \lambda_k}, \tag{2.15}$$

where b_k are the residues of $F(s)$ at the points $-\lambda_k$. On any closed set G that does not contain $s = 0$, the series (2.15) becomes uniformly convergent, if its finitely many terms with poles in G are dropped.

Proof. Consider the auxiliary function

$$\Phi(w) = F\left(\frac{1}{w}\right), \quad \lim_{w\to 0} \Phi(0) = \lim_{s\to\infty} F(s) = 0, \quad \lim_{w\to\infty} \Phi(w) = F(0).$$

The function $\Phi(w)$ is analytic on the complex plane except at the simple poles $w_{-1} = -\frac{1}{B_\alpha}$, $w_k = -\frac{1}{\lambda_k}$, $k = 0, 1, 2 \ldots$. By the Mittag–Leffler theorem (see, for instance, [12]), $\Phi(w)$ can be represented in the form

$$\Phi(w) = \sum_{k=-1}^{\infty} \left(\frac{c_k}{w - w_k} - p_k \right) + c,$$

where c is a constant and c_k are the residues of $\Phi(w)$, and this series becomes uniformly convergent on any closed bounded set, if its terms with poles in that set are dropped. Then

$$\begin{aligned} F(s) &= \sum_{k=-1}^{\infty} \left(\frac{c_k}{\frac{1}{s} + \frac{1}{\lambda_k}} - p_k \right) + c = \sum_{k=-1}^{\infty} \left(\frac{c_k \lambda_k s}{s + \lambda_k} - p_k \right) + c = \\ &= \sum_{k=-1}^{\infty} \left(\frac{c_k \lambda_k^2}{s + \lambda_k} + c_k \lambda_k - p_k \right) + c, \end{aligned}$$

and since $\Phi(0) = 0$, we have

$$F(s) = \sum_{k=-1}^{\infty} \frac{c_k \lambda_k^2}{s + \lambda_k} .$$

Letting $c_k \lambda_k^2 = b_k$, we obtain (2.15).

From Lemma 1 and the uniform convergence of (2.15) in the domain $\operatorname{Re} s \geq s_0 > 0$, we see that for the calculation of the original function $f(t)$ it suffices to apply the inverse Laplace transformation to the series (2.15) term-by-term. Thus, we get

$$f(t) = L^{-1}_{s\to t}[F(s)](t) = \sum_{k=-1}^{\infty} b_k L^{-1}_{s\to t} \left[\frac{1}{s - \lambda_k} \right](t),$$

and finally,

$$f(t) = \sum_{k=-1}^{\infty} b_k \exp[-\lambda_k t], \tag{2.16}$$

where

$$b_{-1} = \frac{1}{J(-B_\alpha)}, \tag{2.17}$$

$$b_k = \frac{1}{(B_\alpha - \lambda_k)} \frac{(1-p^{-1})^{-1}}{\sum_{n=0}^{\infty} \frac{p^{-n}}{(\lambda_k - p^{-\alpha n})^2}}, \qquad k = 0, 1, 2, \ldots. \tag{2.18}$$

It is not difficult to see that the series $\sum_{k=0}^{\infty} b_k$ is convergent and majorizes the series (2.16), which implies uniform convergence of the latter and the continuity of $f(t)$.

The above results can be summed up as follows:

Theorem 2. *The distribution density for the first passage times of a trajectory of the ultrametric random walk can be represented as a uniformly convergent series* (2.16) *whose coefficients are defined by* (2.17) *and* (2.18)*.*

Let us go on with the examination of $f(t)$. It is not difficult to show that

1) $\lim\limits_{t\to 0} f(t) = \lim\limits_{s\to\infty} sF(s) = 0$;

2) $\lim\limits_{t\to\infty} f(t) = \lim\limits_{\substack{s\to 0\\ \operatorname{Re} s>0}} sF(s) = 0$.

Then, since the function $f(t)$ is positive and continuous, it must have a maximum. Let us show that this maximum is unique.

From the first limit, we have $\sum_{k=-1}^{\infty} b_k = 0$. Thus, the series (2.16) can be represented as the difference of two monotonically decreasing strictly concave down functions, $f(t) = -f_1(t) + f_2(t) \geq 0$, and therefore, $\sup\limits_{t\in R_+} |f_1(t) - f_2(t)|$ is unique.

The asymptotic behavior of the function $f(t)$ for all α is described by the following theorem.

Theorema 3. *For the first passage time distribution density $f(t)$ the following estimates hold*:

$$A(\alpha)t^{-\frac{2\alpha-1}{\alpha}}(1+o(1)) \leq f(t) \leq B(\alpha)t^{-\frac{2\alpha-1}{\alpha}}(1+o(1)) \quad \text{for} \quad \alpha > 1; \tag{2.19}$$

$$A(\alpha)t^{-\frac{1}{\alpha}}(1+o(1)) \leq f(t) \leq B(\alpha)t^{-\frac{1}{\alpha}}(1+o(1)) \quad \text{for} \quad \alpha < 1; \tag{2.20}$$

$$A(\alpha)\frac{t^{-1}}{(\ln t)^2}(1+o(1)) \leq f(t) \leq B(\alpha)\frac{t^{-1}}{(\ln t)^2}(1+o(1)) \quad \text{for} \quad \alpha = 1, \tag{2.21}$$

where $o(1) \to 0$ as $t \to \infty$, and $A(\alpha)$, $B(\alpha)$ are functions of α and p.

This theorem is proved in Appendix B.

3 p-Adic Analogue of the Diffusion Equation with Absorbing Region for the First Passage Problem

For the classical problem of random walk of a particle on a straight line, it is well-known that the distribution density function for the first instant at which the particle reaches a given domain can be found from the solution of the diffusion equation with an absorbing region (see, for instance, [13]). We are going to show that a similar approach can be

used in the p-adic case: the distribution density function $f(t)$ for the time of the first return to the domain Z_p can be obtained from the solution of the Cauchy problem for the ultrametric diffusion equation with the absorbing region Z_p, i.e., the equation

$$\frac{\partial\psi(x,t)}{\partial t} = -\frac{1}{\Gamma_p(-\alpha)}\left(\int_{Q_p}\frac{\psi(y,t)-\psi(x,t)}{|x-y|_p^{\alpha+1}}d_py - \Omega(|x|_p)\int_{Q_p\setminus Z_p}\frac{\psi(y,t)}{|x-y|^{\alpha+1}}d_py\right), \quad (3.1)$$

with the initial condition $\psi(x,0) = \Omega(|x|_p)$. The second term in the right-hand side of equation (3.1) is equal to the probability of transition from the region $Q_p\setminus Z_p$ to the absorbing region Z_p per unit time. Since this transition for all trajectories of the random walk (3.1) is always the first one, it follows that the probability density of this passage at the instant t is defined by the formula

$$f(t) = -\frac{1}{\Gamma_p(-\alpha)}\int_{Q_p\setminus Z_p}\frac{\psi(x,t)}{|x|_p^{\alpha+1}}dx. \quad (3.2)$$

Thus, we have two approaches to finding a solution of the first passage problem. Their equivalence is established by the following theorem.

Теорема 4. *The first passage time distribution density function obtained from the solution of the Cauchy problem for the ultrametric diffusion equation with the absorbing region Z_p coincides with the solution of the Volterra equation* (2.10)

Proof: Let us apply the Fourier transformation to $\psi(x,t)$ with respect to the p-adic variable x and then the Laplace transformation with respect to the real variable t. Denote the resulting Fourier–Laplace transform by $\tilde{\Psi}(k,s)$. From (3.1), taking into account the initial condition $\psi(x,0) = \Omega(|x|_p)$, we obtain the following nonhomogeneous Fredholm equation for $\tilde{\Psi}(k,s)$:

$$s\tilde{\Psi}(k,s) = \Omega(|k|_p) - |k|_p^\alpha\tilde{\Psi}(k,s) - \Omega(|k|_p)\int_{Q_p}\tilde{\Psi}(q,s)\left(B_\alpha - |q|_p^\alpha\right)\Omega(|q|_p)d_pq,$$

or

$$\tilde{\Psi}(k,s) = \frac{\Omega(|k|_p)}{s+|k|_p^\alpha} - \frac{\Omega(|k|_p)}{s+|k|_p^\alpha}\int_{Q_p}\tilde{\Psi}(q,s)\left(B_\alpha - |q|_p^\alpha\right)\Omega(|q|_p)d_pq. \quad (3.3)$$

Multiplying equation (3.3) by $\left(B_\alpha - |k|_p^\alpha\right)\Omega\left(|k|_p\right)$ and integrating the result, we get

$$\int_{Q_p}\tilde{\Psi}(k,s)\left(B_\alpha - |k|_p^\alpha\right)\Omega(|k|_p)d_pk = \int_{Q_p}\Omega(|k|_p)\frac{B_\alpha-|k|_p^\alpha}{s+|k|_p^\alpha}d_pk + \\ + \int_{Q_p}\Omega(|k|_p)\frac{B_\alpha-|k|_p^\alpha}{s+|k|_p^\alpha}d_pk\int_{Q_p}\tilde{\Psi}(q,s)\left(B_\alpha - |q|_p^\alpha\right)\Omega(|q|_p)d_pq. \quad (3.4)$$

Note that $\int_{Q_p}\tilde{\Psi}(q,s)\left(B_\alpha - |q|_p^\alpha\right)\Omega(|q|_p)d_pq \equiv F(s)$ is the Laplace transform of the first passage time distribution density function $f(t)$ defined by (3.2). Now, in view of (2.12), we can rewrite equation (3.4) in the form

$$F(s) = G(s) + G(s)F(s).$$

Comparing this with (2.11), we see that the solution of the last equation coincides with that of the Volterra equation (2.10).

4 Number of Returns for Ultrametric Diffusion

In this section, we consider some questions pertaining to the probability of the m-th return on the time interval $(0, t]$ and the growth of the number of returns with the growth of t.

For the probability space $\{\Omega, \Sigma, \mathrm{P}\}$, consider a random process

$$N_{Z_p}(t, \omega) : \Omega \times R^1_+ \to N \subset Z_+, \qquad N_{Z_p}(0, \omega) = 0$$

that describes the number of returns of a particle to the domain Z_p on a finite time interval $(0, t]$. Let us calculate the probability of the m-th return of a particle to Z_p on the interval $(0, t]$. Consider the event $Q^m_t = \{\omega \in \Omega : \ N_{Z_p}(t, \omega) \geq m\}$ that consists in that a particle staying in the domain $Q_p \backslash Z_p$ goes back to Z_p for the m-th time at an instant from the interval $(0, t]$, or equivalently, that a particle visits the domain Z_p at least m times on the time interval $(0, t]$. Denote the probability of this event by $\mathrm{P}\{Q^m_t\} = q^{(m)}(t)$. Obviously, $Q^m_0 = \emptyset$ for all $m > 0$ and $Q^0_t = \Omega$ for all $t > 0$.

Теорема 5. *The probability $q^{(m)}(t)$ of the m-th return on the interval $(0, t]$ satisfies the recurrent equation*

$$\begin{aligned} q^{(m)}(t) &= \int_0^t q^{(m-1)}(t-\tau) f(\tau) d\tau\,, \quad m \geq 1\,, \\ q^{(0)}(t) &= 1, \quad m = 0, \end{aligned} \tag{4.1}$$

where $f(t)$ is the distribution density for the first return time.

Proof. This statement is proved along the same lines as Theorem 1, and therefore, we just outline the main steps.

Consider the event Q^m_t. Let $B_{\tau+d\tau} = \{\omega \in \Omega : \ \tau < \tau_{Z_p}(\omega) \leq \tau + d\tau\}$ be the event of the first return to the domain Z_p on the time interval $(\tau, \ \tau + d\tau]$. Then, $Q^m_t \subset \bigcup_{\tau \in (0,t]} B_{\tau+d\tau}$ and for $m \neq 0$ we can write

$$Q^m_t = Q^m_t \bigcap \left(\bigcup_{\tau \in (o,t]} B_{\tau+d\tau} \right) = \bigcup_{\tau \in (0,t]} \left(Q^m_t \bigcap B_{\tau+d\tau} \right).$$

Since the events $B_{\tau+d\tau}$ are incompatible for all $\tau \in (0, t]$, we have

$$\mathrm{P}\{Q^m_t\} = \sum_{\tau \in (0,t]} \mathrm{P}\{Q^m_t \bigcap B_{\tau+d\tau}\} = \sum_{\tau \in (0,t]} \mathrm{P}\{B_{\tau+d\tau}\} \mathrm{P}\{Q^m_t | B_{\tau+d\tau}\}.$$

Observing that $\mathrm{P}\{Q^m_t \, | B_{\tau+d\tau}\} = \mathrm{P}\{Q^{m-1}_{t-\tau}\}$, we obtain

$$\begin{aligned} \mathrm{P}\{Q^m_t\} &= \sum_{\tau \in (0,t]} \mathrm{P}\{B_{\tau+d\tau}\} \mathrm{P}\{Q^{m-1}_{t-\tau}\}, \quad m \geq 1, \\ \mathrm{P}\{Q^0_t\} &= 1\,, \quad m = 0. \end{aligned} \tag{4.2}$$

Finally, recalling that $\mathrm{P}\{Q^m_t\} = q^{(m)}(t)$ and using the symbolic formula $\mathrm{P}\{B_{\tau+d\tau}\} = f(\tau) d\tau$ (its meaning is clear from the rigorous arguments in the proof of Theorem 1), we obtain the desired recurrent relation (4.1).

For the Laplace transforms, equation (4.1) reads

$$
\begin{aligned}
&Q^{(m)}(s) = Q^{(m-1)}(s)F(s),\\
&Q^{(0)}(s) = s^{-1},
\end{aligned}
\tag{4.3}
$$

where $Q^{(m)}(s)$ is the Laplace transform of $q^{(m)}(t)$.

The solution of (4.3) has the form

$$
Q^{m}(s) = \frac{1}{s}\left(F(s)\right)^{m}, \tag{4.4}
$$

where $F(s)$ is defined by (2.14).

Using the recurrent equation (4.1) and the properties of $f(t)$ and $F(s)$, it is not difficult to show that the functions $q^{(m)}(t)$ have the following properties:

1. Each $q^{(m)}(t)$ is a monotonically increasing function of t; the function $\frac{d}{dt}q^{(m)}(t) \geq 0$ has the meaning of probability density for the m-th return.

2. $\lim\limits_{t\to 0}\dfrac{d}{dt}q^{(m)}(t) = 0, \quad \lim\limits_{t\to\infty}\dfrac{d}{dt}q^{(m)}(t) = 0.$

3. $\lim\limits_{t\to 0} q^{(m)}(t) = 0.$

4. $\lim\limits_{t\to\infty} q^{(m)}(t) = \begin{cases} 1, & \alpha \geq 1,\\ (C_\alpha)^m, & \alpha < 1.\end{cases}$ Recall that the quantity $C_\alpha = \frac{p}{p^\alpha}\left(\frac{p^\alpha - 1}{p-1}\right)^2 < 1$ (see Section 2, property 2 of $f(t)$) is the measure of return trajectories of the ultrametric random walk, which is the same as the probability of the first return on the infinite time interval.

5. $q^{(m)}(t) < q^{(m-1)}(t)$, $m \geq 1$, i.e., the sequence $q^{(m)}(t)$ is monotonically increasing with respect to m for any t.

With the help of the recurrent equation (4.1) and properties 1, 2 of the function $q^{(m)}(t)$, it is not difficult to show that the return probability densities have a maximum, which is unique, and thus, we have a single-mode distribution.

In view of property 4, it is only for $\alpha \geq 1$ that one can speak about the mean waiting time of the m-th return. For $\alpha \geq 1$, the m-th (in particular, the first) return is a certain event, but its mean waiting time is infinite. Indeed,

$$
\int_0^\infty t\frac{d}{dt}q^{(m)}(t)dt = \lim_{\substack{s\to 0\\ \mathrm{Re}\, s>0}}\left(-m\left(F(s)\right)^{m-1}\frac{d}{ds}F(s)\right) = \infty,
$$

since $\lim\limits_{s\to 0,\,\mathrm{Re}\, s>0}\frac{d}{ds}F(s) = -\infty$.

Next, consider the problem of finding the probability of precisely m returns on the time interval $(0,t]$. Let $H_t^m = \{\omega \in \Omega :\ N_{Z_p}(t,\omega) = m\}$ be the event that on the time interval $(0,t]$, the particle goes back to the region Z_p precisely m-times. We are interested in the probability of this event, $\mathrm{P}\{H_t^m\} = h^{(m)}(t)$.

Теорема 6. *The probability $h^{(m)}(t)$ of precisely m returns on the time interval $(0,t]$ satisfies the following recurrent equations:*

$$
\begin{aligned}
h^{(m)}(t) &= \int_0^t h^{(m-1)}(t-\tau)f(\tau)d\tau\,, \quad m \geq 1,\\
h^{(0)}(t) &= 1 - \int_0^t f(\tau)d\tau\,, \quad m = 0\,.
\end{aligned}
\tag{4.5}
$$

where $f(t)$ is the probability density for the first return time.

Proof: We obviously have $H_t^m = Q_t^m \backslash Q_t^{m+1}$, and therefore,

$$h^{(m)}(t) = q^{(m)}(t) - q^{(m+1)}(t). \tag{4.6}$$

Substituting (4.1) into (4.6), we obtain the recurrent equations (4.5). The theorem is proved.

Let us examine more closely the probability distribution function for precisely m returns. In terms of Laplace transforms, equation (4.5) has the form

$$\begin{aligned} H^{(m)}(s) &= H^{(m-1)}(s)F(s), \\ H^{(0)}(s) &= s^{-1}\left(1 - F(s)\right), \end{aligned} \tag{4.7}$$

where $H^{(m)}(s)$ is the Laplace transform of $h^{(m)}(t)$. From (4.7), we obtain the following expression for the Laplace transform of the solution of equation (4.5):

$$H^{(m)}(s) = \frac{1}{s}\left(1 - F(s)\right)\left(F(s)\right)^m. \tag{4.8}$$

Using (2.14) and the solution (4.8), it is not difficult to establish the following properties of $h^{(m)}(t)$:

1. $h^{(m)}(t)$ is a positive function such that
$$\lim_{\substack{t\to 0\\ t\to\infty}} h^{(m)}(t) = \lim_{\substack{s\to\infty\\ s\to 0}} \left(1 - F(s)\right)\left(F(s)\right)^m = 0, \quad \alpha \geq 1;$$
2. $\lim\limits_{t\to 0} h^{(m)}(t) = 0, \quad \alpha < 1, \quad \lim\limits_{t\to\infty} h^{(m)}(t) = \left(1 - C_\alpha\right)\left(C_\alpha\right)^m \quad , \quad \alpha < 1$;
3. $h^{(m)}(t)$ has a maximum, which is unique;
4. $h^{(m)}(t) < h^{(m-1)}(t)$.

What is the mean number of returns $\mu(t)$ on the time interval $(0,t]$? Usually, it is expected that the mean number of returns is proportional to the walk time. By definition, we have

$$\mu(t) = \sum_{n=1}^{\infty} n h^{(n)}(t). \tag{4.9}$$

Theorem 7. *The mean number of returns on the time interval $(0,t]$ is determined by the formula*

$$\mu(t) = \int_0^t g(\tau)d\tau, \tag{4.10}$$

where $g(t)$ is defined by (2.8) and is the density of the probability to return to the domain Z_p at the instant t.

Proof. Writing the expression (4.9) for Laplace transforms and using (4.8), we obtain

$$\mathrm{M}(s) = \sum_{n=1}^{\infty} n H^{(n)}(s) = \frac{1}{s}\left(1 - F(s)\right)\sum_{n=1}^{\infty} n\left(F(s)\right)^n. \tag{4.11}$$

Since $|F(s)| < 1$ for $\mathrm{Re}\, s > 0$, the series in (4.11) can be summed and we have

$$\mathrm{M}(s) = \frac{1}{s}\frac{F(s)}{1-F(s)}\ . \tag{4.12}$$

Hence, using (2.11), we get

$$\mathrm{M}(s) = \frac{1}{s}G(s). \tag{4.13}$$

Applying the inverse Laplace transformation, we obtain (4.10).

Now, let us calculate the average number of returns on the time interval $(0, t]$, using (4.10). Integrating equation (2.3) over the domain Z_p and taking into account (2.8), we obtain the following equation:

$$\frac{\partial}{\partial t}S_{Z_p}(t) = -B_\alpha S_{Z_p}(t) + g(t), \tag{4.14}$$

where $S_{Z_p}(t) = \int_{Z_p}\varphi(x,t)dx$. From the solution of the Cauchy problem for equation (2.3) and the initial condition (2.4), we have the following expression for $S_{Z_p}(t)$:

$$S_{Z_p}(t) = \left(1-\frac{1}{p}\right)\sum_{n=0}^{\infty} p^{-n}\exp\left[-p^{-\alpha n}t\right]. \tag{4.15}$$

The series (4.15) is uniformly convergent, and therefore, using (4.14), (4.15) in (4.10), we easily obtain

$$\mu(t) = B_\alpha\left(1-\frac{1}{p}\right)\sum_{n=0}^{\infty} p^{(\alpha-1)n}\left(1-\exp\left[-p^{-\alpha n}t\right]\right) + \\ + \left(1-\frac{1}{p}\right)\sum_{n=0}^{\infty} p^{-n}\exp\left[-p^{-\alpha n}t\right] - 1. \tag{4.16}$$

Note that the first series in (4.16) is convergent for all $\alpha > 0$, although there is no uniform convergence for $\alpha \geq 1$. The asymptotic behavior of the second series is characterized by the function $t^{-\frac{1}{\alpha}}$ (see formula (A.2) in Apendix A).

Theorem 8. *The following asymptotic estimates hold for the function* $\mu(t)$:

$$p^{-1}\left(1-\frac{1}{p}\right)\Gamma\left(\frac{1}{\alpha}\right)\frac{B_\alpha}{(\alpha-1)\ln p}t^{\frac{\alpha-1}{\alpha}} + O(t^{-\frac{1}{\alpha}}) \leq \\ \leq \mu(t) \ \leq p\left(1-\frac{1}{p}\right)\Gamma\left(\frac{1}{\alpha}\right)\frac{B_\alpha}{(\alpha-1)\ln p}t^{\frac{\alpha-1}{\alpha}} + O(t^{-\frac{1}{\alpha}}), \quad \alpha > 1, \tag{4.17}$$

$$p^{-1}\left(1-\frac{1}{p}\right)\frac{B_1}{\ln p}\Gamma(1)\ln t + O(t^{-1}) \ \leq \ \mu(t) \ \leq \\ \leq p\left(1-\frac{1}{p}\right)\frac{B_1}{\ln p}\Gamma(1)\ln t + O(t^{-1}), \qquad \alpha = 1, \tag{4.18}$$

$$\frac{C_\alpha}{1-C_\alpha} - p^{-(1-\alpha)}\left(1-\frac{1}{p}\right)\Gamma\left(\frac{1-\alpha}{\alpha}\right)\frac{B_\alpha}{\alpha\ln p}t^{-\frac{1-\alpha}{\alpha}}(1+O(t^{-1})) \leq \mu(t) \leq \\ \leq \frac{C_\alpha}{1-C_\alpha} - p^{1-\alpha}\left(1-\frac{1}{p}\right)\Gamma\left(\frac{1-\alpha}{\alpha}\right)\frac{B_\alpha}{\alpha\ln p}t^{-\frac{1-\alpha}{\alpha}}(1+O(t^{-1})), \tag{4.19} \\ 0 < \alpha < 1,$$

where $\Gamma(x)$ *is the gamma-function and* $C_\alpha = \frac{p}{p^\alpha}\left(\frac{p^\alpha-1}{p-1}\right)^2$ *is the probability of the first return on the infinite time interval.*

To obtain the asymptotic estimates of $\mu(t)$ for $\alpha \geq 1$, it suffices to integrate the asymptotic estimates

$$p^{-1}(1-p^{-1})B_\alpha\frac{\Gamma(\alpha^{-1})}{\alpha\ln p}t^{-\frac{1}{\alpha}} \leq g(t) \leq p(1-p^{-1})B_\alpha\frac{\Gamma(\alpha^{-1})}{\alpha\ln p}t^{-\frac{1}{\alpha}}$$

obtained from an asymptotic estimate for the series $S(t)$ (see formula (A.2) in Appendix A). This integration is justified, since the function $g(t)$ and its asymptotic bounds continuously depend on $t \geq a$, $\alpha \geq 1$, are strictly positive for large t, and

$$\int_a^\infty t^{-\frac{1}{\alpha}}dt = +\infty.$$

To obtain the asymptotic estimates of $\mu(t)$ for $0 < \alpha < 1$, it suffices to rewrite (4.16) in the form

$$\begin{aligned}
\mu(t) &= B_\alpha\left(1-\frac{1}{p}\right)\sum_{n=0}^\infty p^{-(1-\alpha)n}\left(1-\exp\left[-p^{-\alpha n}t\right]\right)+ \\
&\qquad + \left(1-\frac{1}{p}\right)\sum_{n=0}^\infty p^{-n}\exp\left[-p^{-\alpha n}t\right]-1 = \\
&= B_\alpha\left(1-\frac{1}{p}\right)\frac{1}{1-p^{-(1-\alpha)}} - 1 - B_\alpha\left(1-\frac{1}{p}\right)\sum_{n=0}^\infty p^{-(1-\alpha)n}\exp\left[-p^{-\alpha n}t\right]+ \\
&\qquad + \left(1-\frac{1}{p}\right)\sum_{n=0}^\infty p^{-n}\exp\left[-p^{-\alpha n}t\right]
\end{aligned}$$

and use formula (A.2) from Appendix A.

Appendix A

Here, we obtain an asymptotic estimate for the series

$$S(t) = \sum_{i=0}^\infty \frac{1}{i^k}a^{-i}e^{-(b)^{-i}t}, \quad t \geq 0,\ k \in N,\ a > 1,\ b > 1, \tag{A.1}$$

for $t \gg 1$.

Lemma A1. *For the series* (A.1) *the following estimate holds for* $t \gg 1$:

$$\begin{aligned}
&(\ln b)^{k-1}(\ln(bt))^{-k}(bt)^{-\frac{\ln a}{\ln b}}\Gamma\left(\frac{\ln a}{\ln b}\right)(1+o(t)) \leq S(t) \leq \\
&\qquad \leq a(\ln b)^{k-1}(\ln(t))^{-k}(t)^{-\frac{\ln a}{\ln b}}\Gamma\left(\frac{\ln a}{\ln b}\right)(1+o(t)),
\end{aligned} \tag{A.2}$$

where $\Gamma(z)$ *is the gamma-function.*

Proof. Note that $\frac{1}{x^k}a^{-x}$ is a decreasing function and $e^{-(b)^{-x}t}$ is an increasing function of x. Therefore, on the interval $i \le x \le i+1$ we have the inequality

$$\frac{1}{x^k}a^{-x}e^{-b^{-(x-1)}t} \le a^{-i}e^{-b^{-i}t} \le \frac{1}{(x-1)^k}a^{-(x-1)}e^{-b^{-x}t}. \tag{A.3}$$

Integrating (A.3) in x from i to $i+1$, we get

$$a^{-1}\int_i^{i+1}\frac{1}{x^k}a^{-(x-1)}e^{-b^{-(x-1)}t}dx \le a^{-i}e^{-b^{-i}t} \le a\int_i^{i+1}\frac{1}{(x-1)^k}a^{-x}e^{-b^{-x}t}dx. \tag{A.4}$$

Now, summing the inequalities (A.4) with respect to i from 0 to ∞, we find that

$$S_{\min}(t) = a^{-1}\int_0^{\infty}\frac{1}{x^k}a^{-(x-1)}e^{-b^{-(x-1)}t}dx \le S(t) \le$$
$$\le a\int_0^{\infty}\frac{1}{(x-1)^k}a^{-x}e^{-b^{-x}t}dx = S_{\max}(t),$$

where

$$S_{\min}(t) = (\ln b)^{k-1}(\ln(bt))^{-k}(bt)^{-\frac{\ln a}{\ln b}}\int_0^{bt}\left(1-\frac{\ln y}{\ln(bt)}\right)^{-k}y^{\frac{\ln a}{\ln b}-1}e^{-y}dy,$$
$$S_{\max}(t) = a(\ln b)^{k-1}(\ln(t))^{-k}(t)^{-\frac{\ln a}{\ln b}}\int_0^{t}\left(1-\frac{\ln y}{\ln(t)}\right)^{-k}y^{\frac{\ln a}{\ln b}-1}e^{-y}dy.$$

Let

$$\int_0^x\left(1-\frac{\ln y}{\ln x}\right)^k y^{z-1}e^{-y}dy = \gamma^{(k)}(z,x)$$

and note that $\lim\limits_{x\to\infty}\gamma^{(k)}(z,x) = \int_0^\infty t^{z-1}e^{-y}dy = \Gamma(z)$ and $\gamma^{(0)}(z,x) = \int_0^x t^{z-1}e^{-y}dy = \gamma(z,x)$. Then, for $x \gg 1$, we can write

$$S_{\min}(t) = (\ln b)^{k-1}(\ln(bt))^{-k}(bt)^{-\frac{\ln a}{\ln b}}\,\Gamma\left(\frac{\ln a}{\ln b}\right)(1+o(1)),$$
$$S_{\max}(t) = a(\ln b)^{k-1}(\ln(t))^{-k}(t)^{-\frac{\ln a}{\ln b}}\,\Gamma\left(\frac{\ln a}{\ln b}\right)(1+o(1)),$$

and therefore, the estimate (A.2) holds.

Appendix B

Proof of Theorem 3 from Section 2

To estimate the function $f(t)$, we first estimate the coefficients b_k of the series (2.16). These coefficients coincide with the residues of the function $F(s)$ at the poles $s = -\lambda_k$, $k = -1, 0, 1, 2, \ldots$ (see (2.17), (2.18)):

$$\begin{aligned}
b_{-1} &= -\frac{1}{J(-B_\alpha)}, \\
b_k &= -\frac{1}{(B_\alpha-\lambda_k)}\lim_{s\to-\lambda_k}\frac{s+\lambda_k}{J(s)} = \frac{1}{(B_\alpha-\lambda_k)}\frac{1}{J'(-\lambda_k)} = \frac{1}{(B_\alpha-\lambda_k)}u_k,
\end{aligned}$$

where

$$u_k = \left(1 - p^{-1}\right)^{-1} \left[\sum_{n=0}^{\infty} \frac{p^{-n}}{(\lambda_k - p^{-\alpha n})^2}\right]^{-1}. \tag{B.1}$$

Recall (see Section 2) that the poles $s = -\lambda_k$, $k = 0, 1, 2, \ldots$ coincide with the simple roots of the equation $\sum_{n=0}^{\infty} p^{-n} \frac{1}{s+p^{-\alpha n}} = 0$ and the values λ_k lie on the interval $p^{-\alpha(k+1)} < \lambda_k < p^{-\alpha k}$. The point $s = 0$ is a limit point for the set of poles. Let us examine the behavior of the poles λ_k and the residues u_k for large k. We pass from λ_k to new variables δ_k, setting

$$\lambda_k = p^{-\alpha k} \left(p^{-\alpha} + (1 - p^{-\alpha})\delta_k\right) = p^{-\alpha(k+1)} + p^{-\alpha k}(1 - p^{-\alpha})\delta_k, \quad 0 \le \delta_k \le 1, \tag{B.2}$$

and let

$$\nu_k = \left(1 - p^{-1}\right) u_k \equiv \left[\sum_{n=0}^{\infty} \frac{p^{-n}}{(p^{-\alpha(k+1)} + p^{-\alpha k}(1 - p^{-\alpha})\delta_k - p^{-\alpha n})^2}\right]^{-1}. \tag{B.3}$$

It can be shown that (B.3) implies the following inequalities for ν_k:

$$\begin{aligned} &\frac{1}{p^{(2\alpha-1)k}} \left[\frac{1 - p^{(2\alpha-1)}}{\left(p^{-\alpha} + (1 - p^{-\alpha})\delta_k\right)^2 \left(1 - p^{2\alpha-1}\right)} + \frac{p}{(1 - p^{-\alpha})^2 \delta_k^2} + \right. \\ &\left. + \frac{p^{-2}}{\left(1 - p^{-\alpha}\right)^2 \left(p^{-\alpha} + (1 - p^{-\alpha})\delta_k\right)^2 \left(1 - p^{-1}\right)}\right]^{-1} < \nu_k < \frac{1}{p^{(2\alpha-1)k}} p(1 - p^{-\alpha})^2 \delta_k^2. \end{aligned} \tag{B.4}$$

Since λ_k, $k = 0, 1, \ldots$, are zeroes of the function $\sum_{n=0}^{\infty} \frac{p^{-n}}{s-p^{-\alpha n}}$, we have

$$\sum_{n=0}^{\infty} \frac{p^{-n}}{p^{-\alpha(k+1)} + p^{-\alpha k}(1 - p^{-\alpha})\delta_k - p^{-\alpha n}} = 0,$$

which implies the following estimate for δ_k:

$$\begin{aligned} &\frac{1}{a_k} \frac{p + 1 + p^{2-\alpha} a_k}{2p^{2-\alpha}} \left[1 - \left(1 - \frac{4p^{2-\alpha} a_k}{\left(p + 1 + p^{2-\alpha} a_k\right)^2}\right)^{1/2}\right] < \\ &\qquad < \delta_k < \frac{1}{a_k} \frac{1 - p^{-\alpha-1}}{(p-1)(1 - p^{-\alpha})}, \end{aligned} \tag{B.5}$$

where $a_k = k$ for $\alpha = 1$ and $a_k = \frac{1-p^{(1-\alpha)(k+1)}}{1-p^{1-\alpha}}$ for $\alpha \ne 1$. The quantities a_k have the following asymptotic behavior for $k \to \infty$:

$$\begin{aligned} a_k &= \frac{1}{1 - p^{1-\alpha}} + o(1) \ \text{ for } \ \alpha > 1, \\ a_k &= \frac{p^{1-\alpha}}{p^{1-\alpha} - 1} p^{(1-\alpha)k} \left(1 + o(1)\right) \ \text{ for } \ \alpha < 1, \\ a_k &= k \ \text{ for } \ \alpha = 1, \quad \text{where } \ o(1) \to 0 \ \text{ as } \ k \to \infty. \end{aligned}$$

Using these relations and (B.4), (B.5), it is not difficult to obtain the estimates

$$D(\alpha)\,(1+o(1)) < \delta_k < U(\alpha)\,(1+o(1))\,,$$
$$\tilde{D}(\alpha)p^{(1-2\alpha)k}\,(1+o(1)) < u_k < \tilde{U}(\alpha)p^{(1-2\alpha)k}\,(1+o(1)) \quad \text{for} \quad \alpha > 1, \tag{B.6}$$

$$D(\alpha)p^{(\alpha-1)k}\,(1+o(1)) < \delta_k < U(\alpha)p^{(\alpha-1)k}\,(1+o(1))\,,$$
$$\tilde{D}(\alpha)p^{-k}\,(1+o(1)) < u_k < \tilde{U}(\alpha)p^{-k}\,(1+o(1)) \quad \text{for} \quad \alpha < 1, \tag{B.7}$$

$$D(\alpha)k^{-1}\,(1+o(1)) < \delta_k < U(\alpha)k^{-1}\,(1+o(1))\,,$$
$$\tilde{D}(\alpha)p^{-k}k^{-2}\,(1+o(1)) < u_k < \tilde{U}(\alpha)p^{-k}k^{-2}\,(1+o(1)) \quad \text{for} \quad \alpha = 1, \tag{B.8}$$

where $D(\alpha)$, $U(\alpha)$, $\tilde{D}(\alpha)$, $\tilde{U}(\alpha)$ are functions of α and p whose expressions are too lengthy to be written out here.

Let $f(t)$ be the probability density function (2.16) for the first passage times. Taking into account the above notation, we can write

$$f(t) = \sum_{k=-1}^{\infty} b_k \exp\left(-\lambda_k t\right) = -\frac{1}{J(-B_\alpha)} \exp\left(-B_\alpha t\right) +$$
$$+\frac{1}{B_\alpha - p^{-\alpha} - (1-p^{-\alpha})\delta_0} u_k \exp\left[-(p^{-\alpha} + (1-p^{-\alpha})\delta_0)t\right] + g(t),$$

where we have set

$$g(t) \equiv \sum_{k=1}^{\infty} \frac{1}{B_\alpha - p^{-\alpha(k+1)} - p^{-\alpha k}(1-p^{-\alpha})\delta_k} u_k \exp\left[-(p^{-\alpha(k+1)} + p^{-\alpha k}(1-p^{-\alpha})\delta_k)t\right].$$

For $g(t)$, we have the estimate

$$\frac{1}{B_\alpha} \sum_{k=1}^{\infty} u_k \exp\left[-p^{-\alpha k} t\right] < g(t) < \frac{1}{B_\alpha - p^{-\alpha}} \sum_{k=1}^{\infty} u_k \exp\left[-p^{-\alpha k} p^{-\alpha} t\right].$$

Further, taking into account (B.6)–(B.8), we find that

$$\frac{\tilde{D}(\alpha)}{B_\alpha} \sum_{k=1}^{\infty} p^{(1-2\alpha)k}\,(1+o(1)) \exp\left[-p^{-\alpha k} t\right] < g(t) <$$
$$< \frac{\tilde{U}(\alpha)}{B_\alpha - p^{-\alpha}} \sum_{k=1}^{\infty} p^{(1-2\alpha)k}\,(1+o(1)) \exp\left[-p^{-\alpha k} p^{-\alpha} t\right], \quad \alpha > 1; \tag{B.9}$$

$$\frac{\tilde{D}(\alpha)}{B_\alpha} \sum_{k=1}^{\infty} p^{-k}\,(1+o(1)) \exp\left[-p^{-\alpha k} t\right] < g(t) <$$
$$< \frac{\tilde{U}(\alpha)}{B_\alpha - p^{-\alpha}} \sum_{k=1}^{\infty} p^{-k}\,(1+o(1)) \exp\left[-p^{-\alpha k} p^{-\alpha} t\right], \quad \alpha < 1; \tag{B.10}$$

$$\frac{\tilde{D}(1)}{B_1} \sum_{k=1}^{\infty} \frac{p^{-k}}{k^2}\,(1+o(1)) \exp\left[-p^{-k} t\right] < g(t) <$$
$$< \frac{\tilde{U}(1)}{B_1 - p^{-1}} \sum_{k=1}^{\infty} \frac{p^{-k}}{k^2}\,(1+o(1)) \exp\left[-p^{-k} p^{-1} t\right], \quad \alpha = 1. \tag{B.11}$$

From (B.9)–(B.11), using the inequalities (A.4), (A.2), we obtain (2.19)–(2.21). The proof is complete.

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References

[1] Ogielski, A.T. and Stein, D.L., Phys. Rev. Lett. 55 (1985) 1634.

[2] Rammal, R., Toulouse, G., and Virasoro, M.A. Ultrametricity for physicists. Rev. Mod. Phys. 589 (1986) 765.

[3] Frauenfelder, H. Complexity in proteins. *Nature Struct. Biol.* **2**, p. 821-823 (1995).

[4] Avetisov,V.A., Bikulov, A.Kh., and Kozyrev, S.V. Application of p-adic analysis to models of spontaneous breaking of replica symmetry, Journal of Physics A, 1999. Vol.32, p. 8785.

[5] Avetisov, V.A., Bikulov, A.Kh., Kozyrev, S.V., and Osipov, V.A. p-Adic Models of Ultrametric Diffusion Constrained by Hierarchical Energy Landscapes. J.Phys. A: Math. Gen., 35 (2002) 177.

[6] Avetisov, V.A., Bikulov, A.Kh., and Osipov, V.A. p-Adic description of characteristic relaxation in complex systems. J. Phys. A: Math. Gen. 36, 2003, 4239.

[7] Avetisov, V.A., Zhuravlev Yu. N. An evolutionary interpretation of the p-adic ultrametric diffusion equation. Doklady Mathematics, **414** p. 309–312.

[8] Becker, O. K., Karplus, M. The topology of multidimentional protein energy surfaces: theory and application to peptide structure and kinetics. J. Chem. Phys. **106** p. 1495–1517 (1997).

[9] Vladimirov, V. S., Volovich, I. V., and Zelenov, E. I. *p-Adic Analysis and Mathematical Physics* Singapure: World Scientific Publishing (1994).

[10] Avetisov, V.A., Bikulov, A.Kh. Protein ultrametricity and spectral diffusion in deeply frozen proteins. (to be pulished, [q-bio]arXive:0804.4551 (2008))

[11] Vladimirov, V. S., *Mathematical Physics Equations*. (in Russian) Moscow: Nauka, 1985.

[12] Lavrentiev, M.A., Shabat, B. V. *Methods of the Theory of Functions in a Complex Variable* (in Russian) Moskow: Nauka, 1987

[13] Van Kampen, N. G. *Stochastic Processes in Physics and Chemistry*. (in Russian) Moscow: Vishaya Shkola, 1990.